A VLFP on Infinite, Finite and Shallow Water

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Diffraction of surface waves by very large floating platform (VLFP) is investigated for three cases: infinite, finite and shallow water. An analytical study is presented for the deflection of the platform, reflection and transmission of incoming waves. For each case the problem is solved for two forms of the platform: strip of the infinite width and semi-infinite plate. The platform is idealized as a plate with elastic properties of zero thickness. Integro-differential, boundary and transition equations and Green's function are used for the solution. Numerical results are obtained for various values of general parameters. Deflection of the semi-infinite platform and of the strip are compared for different values of depth.

1 Introduction

The study of the behavior of floating flexible plates on waves obtain great interest. This problem is important thanks to the investigation of the interaction between large floating platforms (airports etc.) or ice fields and surface waves. The thickness of the floating objects compared to horizontal parameters is small and they are modeled as thin elastic plates.

Here we study the diffraction of surface waves by large floating flexible platform (FFP) of general geometric form which floats on surface of the incompressible fluid of infinite (IWD), finite (FWD) and shallow (SWD) water depth.

We solve the problem for oblique incident waves including perpendicular waves. Two cases of the problem are considered for different form of the platform: a semi-infinite plate and an infinitely long strip of finite width. For both cases results are obtained and compared. Reflection and transmission of incoming waves are investigated.

2 Formulation of the problem

The mathematical formulation is derived for the diffraction of waves by FFP which floats at the surface of an ideal incompressible fluid of depth h which is varied for different cases. Differences of IWD, FWD and SWD cases will be indicated in paper. Incoming short waves propagates from the open fluid (in positive x-direction). We assume waves in still water and introduce the velocity potential $\nabla\Phi(x,y,z,t) = V(x,y,z,t)$. $\Phi(x,y,z,t)$ is a solution of the Laplace equation

$$\Delta\Phi = 0 \tag{1}$$

in the fluid ($z < 0$) together with the conditions:
at the bottom $z = -h$ *(not valid for IWD)*

$$\frac{\partial\Phi}{\partial z} = 0 \tag{2}$$

and surface conditions at $z = 0$

$$\frac{\partial\Phi}{\partial z} = \frac{\partial w}{\partial t} \text{ when } x \in \mathcal{P} \text{ and } \frac{\partial\Phi}{\partial z} = -\frac{1}{g}\frac{\partial^2\Phi}{\partial t^2} \text{ when } x \in \mathcal{F} \tag{3}$$

where $w(x,y,z,t)$ denotes the free surface elevation under the platform. Here and below for case of the strip of width l of infinite length $0 \le x \le l$, $-\infty < y < \infty$ we define fluid area $-\infty < 0 \cup l < \infty$ as $\mathcal{F}$, platform area $0 < x < l$ as $\mathcal{P}$ and the dividing surface $x = 0 \cup x = l$ as $\mathcal{S}$; and for the semi-infinite platform (SIP) $0 \le x < \infty$, $-\infty < y < \infty$ respectively $\mathcal{F}$ is $x < -\infty$, $\mathcal{P}$ is $x > \infty$ and $\mathcal{S}$ is $x = 0$. The platform is assumed to be a thin layer at the free surface $z = 0$, which seems to be a good model for a shallow draft platform which is modeled then as an elastic plate with zero thickness. To describe the deflection of the platform w we apply the thin plate theory, which leads to a differential equation in the following form:

$$D\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 w + m\frac{\partial^2 w}{\partial t^2} = P(x,y,z,t) \tag{4}$$

at $z = 0$ for the platform area $x \in \mathcal{P}$, where m is the mass of unit area of the platform, D is its equivalent flexural rigidity, P is the pressure

$$P(x,y,z,t) = \rho\frac{\partial\Phi}{\partial t} - \rho g w \text{ at } z = 0 \tag{5}$$

here ρ is the density of the water.

The free edges of the platform are free of moment and shear force, boundary conditions are:

$$\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2} = 0 \text{ and } \frac{\partial^3 w}{\partial x^3} + (2-\nu)\frac{\partial^3 w}{\partial x \partial y^2} = 0 \text{ when } x, y \in \mathcal{S} \tag{6}$$

where ν is Poisson's ratio.

For shallow water transition conditions at the edges of the platform are

$$\Phi_n \text{ and } \Phi_t \text{ continuous at } x \in \mathcal{S} \tag{7}$$

where n is the normal to the edge of platform. Physically these conditions express that the mass of the water is conserved and the energy flux is continuous.

The incident wave equals

$$\phi^{inc} = -\frac{ig\zeta}{\omega} e^{ik_0(x\cos\beta + y\sin\beta)} \;;\; \phi^{inc} = -\frac{\cosh k_0(z+h)}{\cosh k_0 h}\frac{ig\zeta}{\omega} e^{ik_0(x\cos\beta + y\sin\beta)} \text{ for FWD} \tag{8}$$

where ζ is the wave height, ω is the frequency and k_0 is the wave number. $k_0 = \omega^2/g$ *for IWD* , $k_0 = \omega/\sqrt{gh}$ *for SWD* while *for FWD* it obeys the dispersion relation $k_0 \tanh k_0 h = K$, here $K = \omega^2/g$. Length of incoming waves is $\lambda = 2\pi/k_0$.

The harmonic wave can be written as

$$\Phi(x,y,z,t) = \phi(x,y,z)e^{-i\omega t} \tag{9}$$

3 Solution

The solution will be derived for all cases for the strip and for the half-plane platform.

For **infinite** and **finite** water depth cases we apply the operator $\partial/\partial t$ to (4) and use the surface condition and (5) to arrive at the following equation for Φ at $z=0$ in the platform area $(x,y) \in \mathcal{P}$:

$$\left\{\frac{D}{\rho g}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 + \frac{m}{\rho g}\frac{\partial^2}{\partial t^2} + 1\right\}\frac{\partial \Phi}{\partial z} + \frac{1}{g}\left\{\frac{\partial^2}{\partial t^2}\right\}\Phi = 0 \tag{10}$$

Using the integro-differential formulation derived in HERMANS (2001) and Green's theorem leads us to such equation for the deflection w:

$$4\pi\left(\mathcal{D}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 - \mu + 1\right)w(x,y) = K\int_{\mathcal{P}}\left(\mathcal{D}\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right)^2 - \mu\right)w(\xi,\eta)\mathcal{G}(x,y,\xi,\eta)\,d\xi d\eta + 4\pi\phi_z^{inc} \tag{11}$$

here we introduce parameters $\mathcal{D} = D/\rho g$, $\mu = m\omega^2/\rho g$. The Green's function, obeying the boundary conditions at the free surface and at the bottom *(for FWD)* and radiation condition, has the form

$$\mathcal{G}(x,y;\xi,\eta) = -2\int_{\mathcal{L}'}\frac{k}{k-k_0}J_0(kr)\,dk \;\text{ for IWD or }\; \mathcal{G}(x,y;\xi,\eta) = -\int_{\mathcal{L}'}\frac{k\cosh kh}{k\sinh kh - K\cosh kh}H_0^1(kr)\,dk \;\text{ for FWD} \tag{12}$$

where $\mathcal{L}'$ - contour of integration in the complex k-plane, $J_0(kr)$ - Bessel's and $H_0^1(kr)$ - Hankel's functions, while r is horizontal distance, so $r = \sqrt{(x-\xi)^2 + (y-\eta)^2}$.

For the platform which floats in **shallow** water in accordance with the derivation of the shallow water theory by STOKER (1957) we have such condition

$$\Phi_z = -h\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\Phi \tag{13}$$

and then from (3) by using of (13) and (9) can be obtained

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi - k_0^2\phi = 0,\; x,y \in \mathcal{F} \tag{14}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi - \frac{i\omega}{h}w = 0,\; x,y \in \mathcal{P} \tag{15}$$

The general solution of (14) and the conditions at ∞ leads to:

$$\phi(x,y) = e^{ik_0(x\cos\beta + y\sin\beta)} + Re^{-ik_0 x\cos\beta + ik_0 y\sin\beta},\; x > l \tag{16}$$

where the first term represent a progressed wave moving to the right and second - reflected wave moving to the left. $|R|$ is amplitude of the reflected wave.

$$\phi(x,y) = Te^{ik_0(x\cos\beta + y\sin\beta)},\; x < 0 \tag{17}$$

where T - the amplitude of the transmitted wave. Values of R and T will be determined.

Insertion of condition (5) to equation (4) and using of (15) leads us to the differential equation for $\phi^{\mathcal{P}}$:

$$\mathcal{D}\Delta^3\phi^{\mathcal{P}} + (1-\mu)\Delta\phi^{\mathcal{P}} - k_0^2\phi^{\mathcal{P}} = 0,\ x,y \in \mathcal{P} \tag{18}$$

which is valid under the platform. For the open fluid surface we have (14).

Transition conditions written in the following form

$$\phi^{\mathcal{P}} = \phi^{\mathcal{F}},\ \frac{\partial\phi^{\mathcal{P}}}{\partial n} = \frac{\partial\phi^{\mathcal{F}}}{\partial n},\ x,y \in \mathcal{S} \tag{19}$$

The deflection of the platform we represents as a superposition of exponential functions in such forms: for the **strip**

$$w(x,y) = \sum_n \big(a_n e^{i\kappa_n x} + b_n e^{-i\kappa_n x}\big) e^{ik_0 y \sin\beta} \tag{20}$$

and for the **semi-infinite** platform

$$w(x,y) = \sum_n a_n e^{i\kappa_n x + ik_0 y \sin\beta} \tag{21}$$

for $0 \le \beta \le \beta_{cr} \le \pi/2$ at $z = 0$ where 'amplitudes' a_n, b_n and reduced wave numbers κ_n will be determined later for angle of incidence smaller than a critical angle, which will be described below. For the description of the connection between platform deflection and potential we use first condition of (3) *for IWD and FWD* and (15) *for SWD*.

We define κ_n as

$$\kappa_n = \sqrt{\kappa^{(n)2} - k_0^2\sin^2\beta} \tag{22}$$

where $\kappa^{(n)}$ are roots of dispersion relation which is varied for different cases:

$$\big(\mathcal{D}\kappa^4 - \mu + 1\big)\kappa = \pm k_0 \text{ for IWD} \tag{23}$$

$$\big(\mathcal{D}\kappa^4 - \mu + 1\big)\kappa\tanh\kappa h = k_0 \text{ for FWD} \tag{24}$$

$$\big(\mathcal{D}\kappa^4 - \mu + 1\big)\kappa^2 = k_0^2 \text{ for SWD} \tag{25}$$

Taking into account for every case 3 roots [1] which are physically realistic solutions for κ and are situated in the upper complex half-plane: positive real root $\kappa^{(1)}$ and two roots in complex plane $\kappa^{(2)}$ and $\kappa^{(3)}$ with equal imaginary parts and equal but opposite-signed real parts. Also we determine a value of critical angle of incidence. Angle becomes critical when κ_1 (corresponded value to real positive root $\kappa^{(1)}$ of dispersion relation) is equal 0. Then

$$\sin\beta_{cr} = \kappa^{(1)}/k_0 \tag{26}$$

Boundary conditions are the same for all values of depth. For the strip we get 4 equations from boundary conditions at both edges $x = 0$ and $x = l$ from zero moment and zero shear force conditions (6). For the semi-infinite platform we have two boundary conditions at the edge $x = 0$.

Determination of 'amplitudes' a_n and b_n. We consider *for IWD and FWD* the zeros of the dispersion relation for the water surface. We insert (20) in (11) and obtain two linear relations:

$$\sum_{n=1}^{3}\big(\mathcal{D}\kappa^{(n)^4} - \mu\big)Ck_0\bigg(\frac{a_n}{(\kappa_n - k_0\cos\beta)\cos\beta} - \frac{b_n}{(\kappa_n + k_0\cos\beta)\cos\beta}\bigg) + \zeta = 0 \tag{27}$$

$$\sum_{n=1}^{3}\big(\mathcal{D}\kappa^{(n)^4} - \mu\big)Ck_0\bigg(-\frac{a_n e^{i\kappa_n l}}{(\kappa_n + k_0\cos\beta)\cos\beta} + \frac{b_n e^{-i\kappa_n l}}{(\kappa_n - k_0\cos\beta)\cos\beta}\bigg) = 0 \tag{28}$$

where $C = 1$ *for IWD* and $C = K/\big(K(1-Kh) + k_0^2 h\big)$ *for FWD*. Together with the boundary conditions we have 6 equations. The solution of this system gives us values of deflection and reflection and transmission coefficients also if we compute contribution of pole in region $x < 0$ and $x > l$ respectively. This is the solution for the strip. For the semi-infinite platform we use (27) without b_n-term and 2 boundary conditions.

Determination of 'amplitudes' *for SWD*. Reflection R and transmission T coefficients will be find at once. Here we need 8 equations, 4 from boundary conditions we have already. Rest 4 equations can be obtained from transition conditions (19) at both edges of the platform. Due to (16) and (17) we have at $x = 0$ and at $x = l$:

$$1 + R = -\frac{\omega}{ih}\sum_{n=1}^{3}(a_n + b_n);\ ik_0\cos\beta(1-R) = -\frac{\omega}{ih}\sum_{n=1}^{3} i\kappa_n(a_n - b_n) \tag{29}$$

$$Te^{ik_0 l\cos\beta} = -\frac{\omega}{ih}\sum_{n=1}^{3}\big(a_n e^{i\kappa_n l} + b_n e^{-i\kappa_n l}\big);\ ik_0\cos\beta Te^{ik_0 l\cos\beta} = -\frac{\omega}{ih}\sum_{n=1}^{3} i\kappa_n\big(a_n e^{i\kappa_n l} - b_n e^{-i\kappa_n l}\big) \tag{30}$$

Such way for the definition of 6 components of the deflection and coefficients of reflection and transmission we have system of 8 equations. Deflection and reflection coefficient for the SIP can be obtained by same way also. From transition condition (19) and (16) we obtain conditions (29) at the edge $x = 0$ but without b_n-terms. After solving of the system we obtain 3 components of the deflection and reflection coefficient for the SIP.

[1] in fact *for FWD* we can use n roots but here we use only three for exact comparison of solutions and results

4 Discussion of Results

We present some numerical results on this chapter and compare them for different cases of the water depth and platforms.

In figure 1 we show comparison of the results for the strip deflection for different wavelengths. Results are shown for three values of depth for zero angle of incidence. In figure 2 we compare the results for the deflection of the strip and of the semi-infinite platform for same wavelength and depth. For each form of the platform results are presented for 4 different values of incident angle. Also we compare reflection and transmission *for SWD* and *for FWD* cases. These results are presented in figure 3, where R_∞ is reflection coefficient for the semi-infinite platform. Wave energy is conserved, up to a high degree of accuracy $|R|^2 + |T|^2 = 1$.

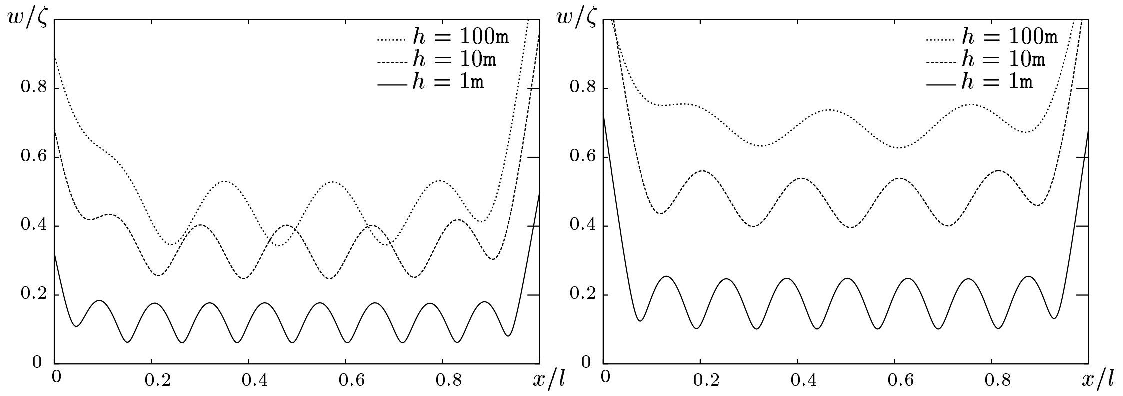

Fig.1 Results for the strip deflection for $D/\rho g = 10^5\mathrm{m}^4$, $h = 1, 10, 100\mathrm{m}$ for $\lambda = 0.3l$ (a) and $\lambda = 0.5l$ (b)

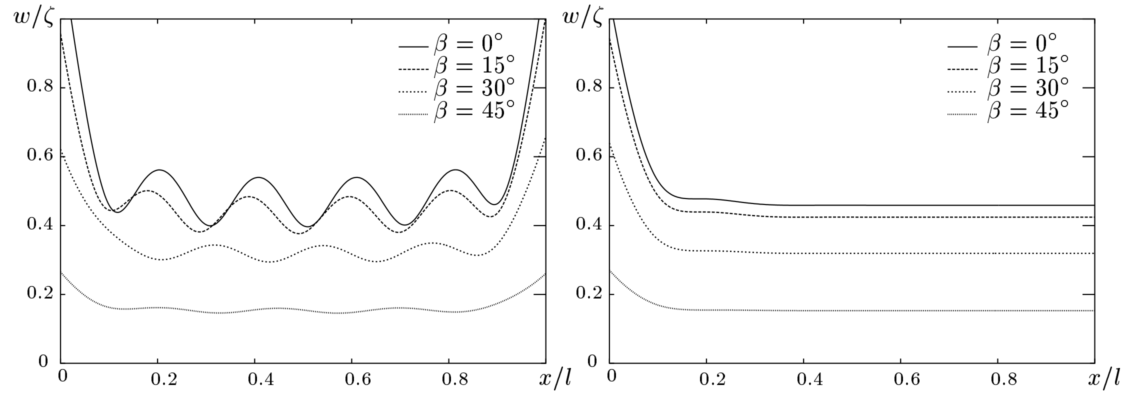

Fig.2 Results for the deflection for $D/\rho g = 10^5\mathrm{m}^4$, $h = 10\mathrm{m}$, $\lambda = 0.5l$ for the strip (a) and for the SIP (b)

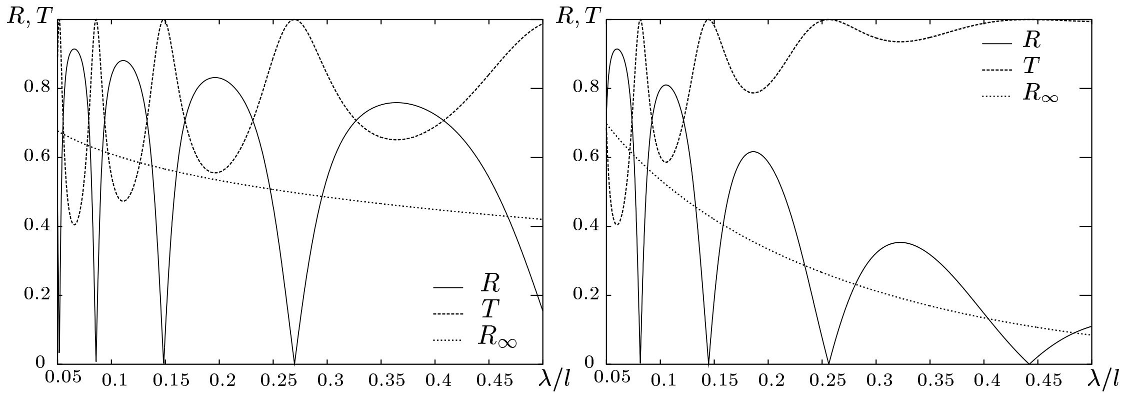

Fig.3 Reflection and transmission coefficients for $D/\rho g = 10^5\mathrm{m}^4$, $\beta = 0°$ for $h = 1\mathrm{m}$ (a) and $h = 100\mathrm{m}$ (b)

References

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