HYDROELASTICITY OF ELASTIC CIRCULAR PLATE

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Summary
In this paper we study the diffraction of incident surface waves on a floating elastic circular plate. The hydroelastic response of the plate to a plane incident wave is investigated. An integro-differential equation is derived for the problem. The free surface elevation and Green's function are expressed in cylindrical coordinates as a superposition of Bessel functions. For the coefficients, a set of algebraic equations is obtained, yielding the approximate solution. First, the case of infinite depth is considered. Then, a solution is obtained for the general case of finite water depth analogously. The exact solution is approximated by taking a finite number of roots of the dispersion relation into account.

Keywords: surface waves, hydroelasticity, diffraction, VLFP, elastic circular plate, deflection, wave modes, integro-differential equation, Green's function, Bessel functions, Hankel functions.

1 Introduction

Recently a number of papers was published considering the hydroelastic response of very large floating platforms. There are several approaches to describe the interaction between VLFP and surface waves, the following can be distinguished: asymptotic theory for short waves [1], parabolic approximation [2], ray theory, variation equation method, eigenfunction expansion method, Galerkin method, Wiener-Hopf technique. Many of these results were presented at previous IWWWFB.

Mainly, the indicated articles studied the plates of one or two infinite dimensions which significantly simplify the complexity of analysis. For plates of finite extent, numerical methods are often used. Here we study this problem analytically. We use Green's theorem and an integro-differential formulation for the deflection as derived and described in [3]. At previous Workshops we presented this approach for plates of semi-infinite [4] or quarter-infinite extent [5].

Here, we consider a circular plate. This problem (the plate modeled an ice field) was solved by Meylan and Squire [6] for deep water. A closed form solution for a buoyant circular plate floating on shallow water was found by Zilman and Miloh [7]. We consider the problem of an elastic circular plate of constant flexural rigidity and homogeneous stiffness for two different cases: deep water and water of finite depth. The plate deflection is generated by incoming surface waves. The edge of the plate is free of shear forces and bending moments. We describe the hydroelastic response of the plate to water surface waves. The plate deflection is represented as a summation of the product of Bessel functions and cosine functions.

At first, we study the behavior of a plate plate floating on the surface of water of infinite depth (IWD). Also, this case is a good starting point to construct a solution for the general case of finite water depth (FWD). The general analysis and set of equations are more complicated for the FWD case as more roots of the water dispersion relation have to be taken into account. Finally, we show and analyze numerical results for various physical parameters of the problem.

2 Formulation of the Problem

In this section, we derive the general mathematical formulation for the titled problem. The floating thin elastic circular plate of radius r_0 covers part of the surface of ideal, incompressible water. The water depth h is infinite for the case of deep water and finite for the other case. We assume that no space (gap) exists between the free surface and the plate. The flexural rigidity of the circular plate is constant.

The plate deflection is generated by surface waves propagating in positive x-direction. The wave amplitude is rather small compared to the wave height and the water depth. We assume that waves propagate in still water. The problem is considered in polar coordinates, but Cartesian coordinates are used to derive main equations. The sketch of the geometry is shown in Figure 1. We divide the fluid domain on the region covered by plate $\mathcal{P}$ and the open fluid (water) region $\mathcal{F}$ with the plate contour $\mathcal{S}$ ($\rho = r_0$) as dividing line.

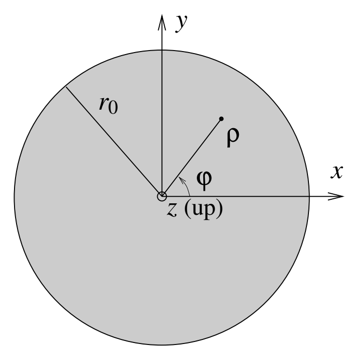

Figure 1: Coordinate system of the problem.

The velocity potential $\Phi(\vec{x},t)$ is a solution of the governing Laplace equation

$$\Delta\Phi = 0 \tag{1}$$

in the fluid ($z < 0$). Equation (1) supplemented with the boundary conditions at the free surface and at the bottom for finite depth model, which are described in [3] and [4]. The linearized free surface condition takes the form at $z = 0$ for $x, y \in \mathcal{F}$

$$\frac{\partial\Phi}{\partial z} = -\frac{1}{g}\frac{\partial^2\Phi}{\partial t^2}, \tag{2}$$

where g is gravitational acceleration and t is the time.

The platform is modeled as an elastic plate with zero thickness. Such model can be applied due to small thickness and shallow draft of the platform. To describe the deflection of the plate $w(x,y,t)$ we apply the isotropic thin plate theory [8]. We apply the operator $\partial/\partial t$ to a standard differential equation and use the surface conditions to arrive at the following equation for Φ on the plate area $\mathcal{P}$ at $z = 0$:

$$\left(\mathcal{D}\Delta^2 + \mu\frac{\partial^2}{\partial t^2} + 1\right)\frac{\partial\Phi}{\partial z} - \frac{1}{g}\frac{\partial^2\Phi}{\partial t^2} = 0, \tag{3}$$

where the parameters $\mathcal{D} = D/\rho_w g$, $\mu = m\omega^2/\rho_w g$ are constant and m is the mass of unit area of the platform, D is its equivalent flexural rigidity, and ρ_w is the density of the water. We consider harmonic waves, so $\Phi(x,y,z,t)=\phi(x,y,z)e^{-i\omega t}$. Then we reduce time-dependence, consider waves of a single frequency ω and obtain

$$\left(\mathcal{D}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2-\mu+1\right)\frac{\partial\phi}{\partial z}-K\phi=0, \tag{4}$$

at $z=0$, where $K=\omega^2/g$. The potential of the undisturbed incident waves in polar coordinates is

$$\phi^{inc}(\rho,\varphi,z)=\begin{cases}\dfrac{gA}{i\omega}e^{ik_0\rho\cos\varphi+k_0z} & \text{for IWD,}\\ \dfrac{\cosh k_0(z+h)}{\cosh k_0h}\dfrac{gA}{i\omega}e^{ik_0\rho\cos\varphi} & \text{for FWD,}\end{cases} \tag{5}$$

where A is the wave height and k_0 is the wave number. For the IWD $k_0=K$, while for the FWD k_0 obeys the water dispersion relation

$$k_0\tanh k_0h=K. \tag{6}$$

The wave length of incoming waves is $\lambda=2\pi/K$. We consider the situation when the wave length is less than the diameter of the plate ($\lambda<2r_0$). The edge of the circular plate is free of shear forces and bending moments and the free edge conditions at the plate contour $\mathcal{S}$ are written as:

$$\left(\nabla^2-\frac{(1-\nu)}{\rho}\left(\frac{\partial}{\partial\rho}+\frac{1}{\rho}\frac{\partial^2}{\partial\varphi^2}\right)\right)w=0, \tag{7}$$

$$\left(\frac{\partial}{\partial\rho}\nabla^2+\frac{(1-\nu)}{\rho^2}\left(\frac{\partial}{\partial\rho}-\frac{1}{\rho}\right)\frac{\partial^2}{\partial\varphi^2}\right)w=0, \tag{8}$$

where ν is Poisson's ratio.

3 Derivation of IDE

Now we have only two equations from the free edge conditions to determine the unknown amplitudes. In this section we will derive an integro-differential equation (IDE) to complete the set of equations that obeys the conditions of the continuity for velocity and potential at the plate contour $\mathcal{S}$. The formulation has been derived in [9] and [3] for the general 3D case and now we will rederive it in polar coordinates for the circular plate.

The total potential in area covered by plate $\mathcal{P}$ is denoted by $\phi^{\mathcal{P}}$, while in the water region $\mathcal{F}$ it is written as a superposition of the incident wave potential ϕ^{inc} and ϕ^{dis}, which is the sum of classical diffraction potential and radiation potential.

We introduce the Green's function $\mathcal{G}(\vec{x},\vec{\xi})$ that fulfills $\Delta\mathcal{G}=4\pi\delta(\vec{x}-\vec{\xi})$, the free surface and the radiation conditions. We apply Green's theorem to the potentials in $\mathcal{F}$ and $\mathcal{P}$ respectively. The Green's function itself has only a weak singularity, so we may take the limit $z\to0$ and use (4) to express $\phi^{\mathcal{P}}$ in terms of an operator acting on $\phi_z^{\mathcal{P}}$. Then we change from potential to deflection function and obtain the general IDE

$$\left(\mathcal{D}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2-\mu+1\right)w(x,y)=$$
$$\frac{K}{4\pi}\int\limits_{\mathcal{P}}\left(\mathcal{D}\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right)^2-\mu\right)\mathcal{G}(\vec{x},\vec{\xi})w(\xi,\eta)\,d\xi d\eta+Ae^{ik_0x} \tag{9}$$

for the plate deflection w at $z=0$. The IDE in general form in polar coordinates at the free surface, derived analogously, takes the form

$$\left(\mathcal{D}\Delta^2-\mu+1\right)w(\rho,\varphi)=$$
$$\frac{K}{4\pi}\int\limits_{\mathcal{P}}\left(\mathcal{D}\Delta^2-\mu\right)\mathcal{G}(r,\theta;\rho,\varphi)w(r,\theta)\,r\,dr\,d\theta+Ae^{ik_0\rho\cos\varphi}, \tag{10}$$

where $\mathcal{G}(r,\theta;\rho,\varphi)$ is the Green's function in polar coordinates, and the last term represents the potential of incoming waves.

4 Deflection & Green's Function

Here we will describe the deflection, Green's function and corresponding Bessel functions for circular plate. The plate deflection can be represented as a series of Bessel functions with corresponding coefficients in the following form:

$$w(\rho,\varphi)=\sum_{m=1}^{M}\sum_{n=0}^{N}a_{mn}J_n(\kappa_m\rho)\cos n\varphi, \tag{11}$$

where a_{mn} are the unknown amplitude functions, κ_m are the reduced wave numbers, and M is the number of roots of the 'plate' dispersion relation taken into account. For the IWD case we will use 3 roots of the dispersion relation. For the FWD case more than three roots need to be taken into account. This is the result of water dispersion relation (6) which has one real root k_0 corresponding to only one wave number of deep water and a number of imaginary roots k_i, $i=1..M-3$.

The Green's function obeying the boundary conditions at the free surface (and at the bottom for FWD) and the radiation condition has the form

$$\mathcal{G}(x,y;\xi,\eta)=-2\int\limits_{\mathcal{L}}F(k)J_0(kR)\,dk, \tag{12}$$

at $z=0$, where

$$F(k)=\begin{cases}\dfrac{k}{k-k_0} & \text{for IWD,}\\ \dfrac{k\cosh kh}{k\sinh kh-K\cosh kh} & \text{for FWD,}\end{cases}$$

$\mathcal{L}$ is the contour of integration in the complex k-plane from 0 to $+\infty$, underneath the singularity $k=k_0$ for IWD or underneath the branch cuts $k=k_i$ for FWD, chosen for satisfying of the radiation condition, $J_0(kR)$ is the Bessel function. Due to Graf's addition theorem it can be replaced by the series

$$J_0(kR)=\sum_{q=0}^{\infty}\delta_qJ_q(kr)J_q(k\rho)C_q, \tag{13}$$

where $\delta_0=1$ and $\delta_q=2$ for $q>0$, r is the distance from the center of the plate to the point of observation, $C_q=\cos q(\theta-\varphi)$ and $\theta-\varphi$ is the angle between r and ρ. The upper limit in (13) can be taken as finite due to decaying behavior of Bessel functions. So, the Green's function for circular plate in polar coordinates takes the form

$$\mathcal{G}(r,\theta;\rho,\varphi)=-2\int\limits_0^{\infty}F(k)\sum_{q=0}^{N}\delta_qJ_q(kr)J_q(k\rho)C_q\,dk. \tag{14}$$

We insert the relations for the deflection (11) and Green's function (14) into (9) and obtain the governing expanded integro-differential equation at the free surface $z=0$

$$(\mathcal{D}\Delta^2-\mu+1)\sum_{m=1}^{M}\sum_{n=0}^{N}a_{mn}J_n(\kappa_m\rho)\cos n\varphi$$
$$+\frac{K}{2\pi}\int\limits_0^{2\pi}\int\limits_0^{r_0}(\mathcal{D}\Delta^2-\mu)\sum_{m=1}^{M}\sum_{n=0}^{N}a_{mn}J_n(\kappa_mr)\cos n\theta\int\limits_0^{\infty}F(k)$$
$$\times\sum_{q=0}^{N}\delta_qJ_q(kr)J_q(k\rho)C_q\,dk\,r\,dr\,d\theta=A\sum_{n=0}^{N}\varepsilon_nJ_n(k_0\rho)\cos n\varphi, \tag{15}$$

where $\varepsilon_n = \delta_n i^n$. Due to the orthogonality relation for the cosine function we only get a non zero contribution for $n = q$. The value of N will be chosen later. First we close the contour of the integration. Then we work out the integration with respect to r and θ in (15) and obtain the following set of $N+1$ equations from our IDE ($n = 0..N$):

$$\sum_{m=1}^{M}(\mathcal{D}\kappa_m^4 - \mu + 1)a_{mn}J_n(\kappa_m\rho) +$$
$$Kr_0\int_0^\infty \sum_{m=1}^{M}(\mathcal{D}\kappa_m^4 - \mu)a_{mn}\frac{F(k)J_n(k\rho)}{(k^2-\kappa_m^2)}\Big[kJ_{n+1}(kr_0)J_n(\kappa_m r_0)$$
$$-\kappa_m J_n(kr_0)J_{n+1}(\kappa_m r_0)\Big]\,dk = A\varepsilon_n J_n(k_0\rho). \qquad (16)$$

5 Infinite Water Depth

We transform the integral in (16) to the integral along vertical axis in complex plane plus a sum of the residues. Application of the residue lemma at the poles $k = \kappa_m$ leads to a dispersion relation if we apply the residue lemma to the integrand with preceding representation of each Bessel function as sum of Hankel functions of first and second kind. Then the Wronskian can be used, see e.g. [10], at the poles $k = \kappa_m$ for the combination of Hankel functions

$$W\{H_n^{(1)}(\kappa_m r_0), H_n^{(2)}(\kappa_m r_0)\} = -\frac{4i}{\pi\kappa_m r_0}. \qquad (17)$$

Thus we arrive at a dispersion relation for deep water

$$\big(\mathcal{D}\kappa^4 - \mu + 1\big)\kappa = \pm k_0. \qquad (18)$$

We take 3 roots of the dispersion relation into account for the current problem: real positive κ_1, complex κ_2 and κ_3 with equal real and equal but opposite-signed imaginary parts. The real root κ_1 represents the main traveling wave, the 2 complex roots represent evanescent waves.

After analysis of the set of $N+1$ integral equations and dispersion relation we have to consider the pole $k = k_0$. We repeat the transform of the integral in (16) and represent each of Bessel functions as a half-sum of Hankel functions. Then, a contour of integration can be closed. If we neglect the integral along the imaginary axis, application of Jordan's lemma and the contribution of the pole $k = k_0$ lead us to $N+1$ relations:

$$2\pi i r_0 \sum_{m=1}^{3}\frac{k_0^2}{k_0^2-\kappa_m^2}\Big[k_0 H_{n+1}^{(1)}(k_0 r_0)J_n(\kappa_m r_0)$$
$$-\kappa_m H_n^{(1)}(k_0 r_0)J_{n+1}(\kappa_m r_0)\Big](\mathcal{D}\kappa_m^4 - \mu)a_{mn} = A\varepsilon_n. \qquad (19)$$

The system for the determination of $3(N+1)$ unknown amplitudes a_{mn} can be completed by $2(N+1)$ equations obtained from the free edge conditions (7) and (8).

6 Finite Water Depth

Now, we consider the more general case when the plate floats on water of finite depth. Let us analyze the meromorphic function $F(k)$. The poles of function $F(k)$ are roots of a dispersion relation for the water region (6) $k = \pm k_i$ ($i = 0..M-3$), where k_0 is the positive real root and k_i (for $i \neq 0$) is the purely imaginary. The meromorphic function $F(k)$ is bounded for all roots and can be described by the following relation:

$$F(k) = \sum_{i=0}^{M-3}\frac{k_i^2}{k_i^2 h - K^2 h + K}\left(\frac{1}{k+k_i} + \frac{1}{k-k_i}\right). \qquad (20)$$

Now we insert relation (20) into (16), where we consider two integrals in the complex k-plane, that can be combined into one integral from $-\infty$ to $+\infty$ with the poles $k = k_i$ only. Finally, we arrive at the governing integro-differential equation for FWD

$$\sum_{m=1}^{M}(\mathcal{D}\kappa_m^4 - \mu + 1)a_{mn}J_n(\kappa_m\rho) + Kr_0\int_{-\infty}^{\infty}\sum_{m=1}^{M}(\mathcal{D}\kappa_m^4 - \mu)a_{mn}$$
$$\times\frac{J_n(k\rho)}{(k^2-\kappa_m^2)}\sum_{i=0}^{M-3}\frac{k_i^2}{(k_i^2 h - K^2 h + K)(k-k_i)}$$
$$\times\big[kJ_{n+1}(kr_0)J_n(\kappa_m r_0) - \kappa_m J_n(kr_0)J_{n+1}(\kappa_m r_0)\big]\,dk$$
$$= A\varepsilon_n J_n(k_0\rho) \qquad (21)$$

at $z = 0$. We use the Wronskian (17) and apply the residue lemma. The poles $k = \kappa_m$ lead to the standard dispersion relation for water of finite depth

$$\big(\mathcal{D}\kappa^4 - \mu + 1\big)\kappa\tanh\kappa h = K. \qquad (22)$$

Here we will use M roots of dispersion relation: one real positive κ_1, two complex κ_2 and κ_3, and $M-3$ imaginary roots. The position of roots κ_m in the complex k-plane is similar to roots of water dispersion relation, except for two complex roots κ_2 and κ_3, which are located in the upper half-plane.

We split up the Bessel functions into

$$J_q(kr_0) = \frac{H_q^{(1)}(kr_0) + H_q^{(2)}(kr_0)}{2}, \qquad (23)$$

where q is n or $n+1$. The integrals with $H_q^{(1)}(kr_0)$ may be closed in the upper-half plane, see Figure 2. The integrals with $H_q^{(2)}(kr_0)$ may be closed in the lower half-plane. In the latter case we get a zero contribution.

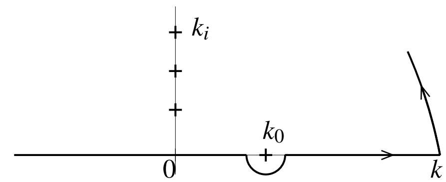

Figure 2: Closure of the contour

Application of Cauchy's theorem to the integral closed in the upper half-plane gives the following $N+1$ equations to determine the amplitudes a_{mn}

$$2\pi i r_0\sum_{m=1}^{M}\frac{Kk_0^2}{(k_0^2-\kappa_m^2)(k_0^2 h - K^2 h + K)}\Big[k_0 H_{n+1}^{(1)}(k_0 r_0)J_n(\kappa_m r_0) -$$
$$\kappa_m H_n^{(1)}(k_0 r_0)J_{n+1}(\kappa_m r_0)\Big](\mathcal{D}\kappa_m^4 - \mu)a_{mn} = A\varepsilon_n, \qquad (24)$$

and the poles $k = k_i$, where $i = 1..M-3$, result in a set of $(M-3)(N+1)$ equations

$$2\pi i r_0\sum_{m=1}^{M}\frac{Kk_i^2}{(k_i^2-\kappa_m^2)(k_i^2 h - K^2 h + K)}\Big[k_i H_{n+1}^{(1)}(k_i r_0)J_n(\kappa_m r_0) -$$
$$\kappa_m H_n^{(1)}(k_i r_0)J_{n+1}(\kappa_m r_0)\Big](\mathcal{D}\kappa_m^4 - \mu)a_{mn} = 0. \qquad (25)$$

Analogously to previous section, the free edge conditions (7) and (8) give us $2(N+1)$ equations. So, we have derived the system of $M(N+1)$ equations to determine the amplitudes a_{mn} for a plate floating in water of finite depth.

7 Numerical Results and Discussion

After solving the system for IWD or FWD case, the deflection of the circular plate can be computed by (11). The amplitudes a_{mn} of each wave mode behave as decaying functions because of the convergence of Bessel functions. Similar behavior of these functions was reported by Zilman and Miloh [7] for shallow water. If we increase the flexural rigidity or the radius, the functions converge faster. Taking first 30 terms of series into account leads to sufficient accuracy for realistic values of the rigidity. So, we define $N = 30$. To avoid difficulties in numerical computation when the argument of the functions is small, it is possible to use the recurrence relation described in [10].

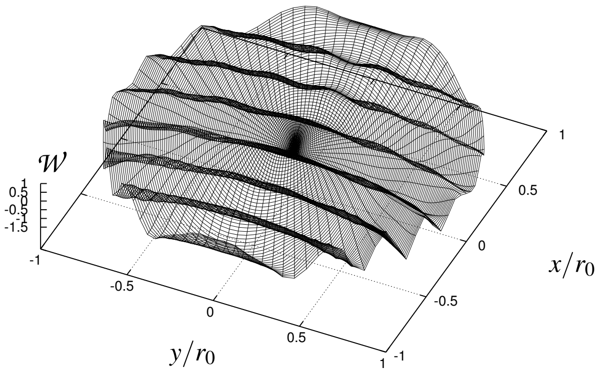

Figure 3: IWD. $r_0 = 500\mathtt{m}$, $\lambda = 100\mathtt{m}$, $\mathcal{D} = 10^5\mathtt{m}^4$.

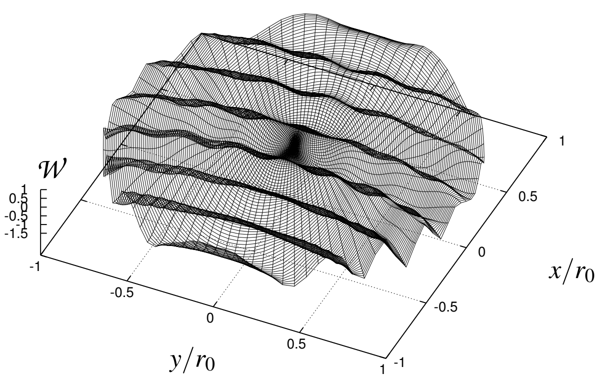

Figure 4: FWD. $h = 100\mathtt{m}$, $r_0 = 500\mathtt{m}$, $\lambda = 100\mathtt{m}$, $\mathcal{D} = 10^5\mathtt{m}^4$.

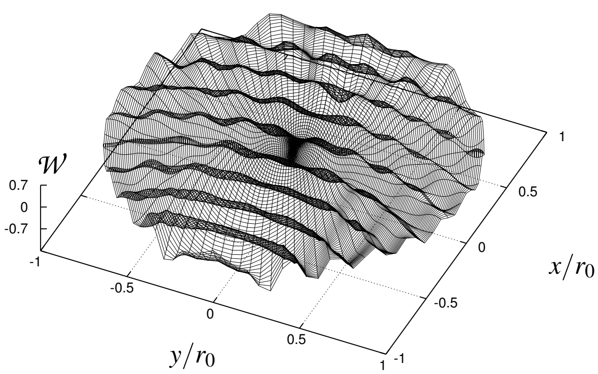

Figure 5: FWD. $h = 100\mathtt{m}$, $r_0 = 500\mathtt{m}$, $\lambda = 50\mathtt{m}$, $\mathcal{D} = 10^5\mathtt{m}^4$.

We show numerical calculations based on various values of plate radius and flexural rigidity, while Poisson's ratio $\nu = 0.25$ and the ratio $m/\rho_w = 0.25\mathtt{m}^3$ are constant. Taking the wave height $A = 1\mathtt{m}$, and varying water depth and incident wave length leads to different values of wave number k_0, and respectively, to the different values of frequency ω.

The number of the roots of a dispersion relation which are taking into account for FWD case is $M = 10$. Generally, any of the imaginary roots κ_m, $m = 4..M$ does not affect the solution much. More details about the number of roots has been published in [3].

In the Figures 3-5 $\mathcal{W}$ denotes the real part of the plate deflection normalized by wave height, i.e. $Re(w)/A$. The figures demonstrate that the wave traveling through plate area propagates with a curved wave front. In the zone closest to the edge of the plate, the deflection has special behavior and can be quite different from the deflection in the center zone, especially for low rigidity of the floating plate. For $\mathcal{D} > 10^9\mathtt{m}^4$ the plate behaves as a very rigid body, whereas for $\mathcal{D} < 10^3\mathtt{m}^4$ the plate hardly has any influence on surface waves. For smaller values of plate rigidity, the plate deflection increases. If the wave length decreases or the water depth increases, then the deflection increases.

8 Conclusions

For the deflection of a thin elastic circular plate floating at the free surface of the water with finite depth an exact analytical solution has been obtained. For infinitely deep water the problem is solved approximately.

The finite water model can be used to solve the problem for water of shallow or infinite depth. Floating platforms should be located in offshore zones. The water depth is rather small in those zones, but as wave length could be both short and long, it is more general to use the finite water depth results to describe the response of the plates to ocean or sea waves.

The presented approach can be extended to other rotational symmetric configurations. The reflection and the transmission coefficients of incoming waves can also be described by this approach. More details and results will be presented at the Workshop and published later in [11].

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